Coordinate System

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Outline

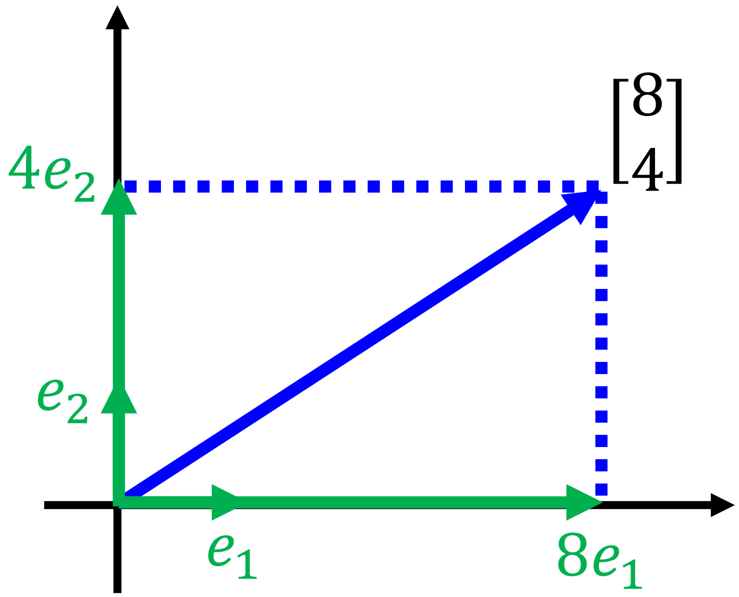
- Coordinate Systems
  - Each coordinate system is a “viewpoint” for vector representation.
    - The same vector is represented differently in different coordinate systems.
    - Different vectors can have the same representation in different coordinate systems.
- Changing Coordinates
- Reference: textbook Ch 4.4

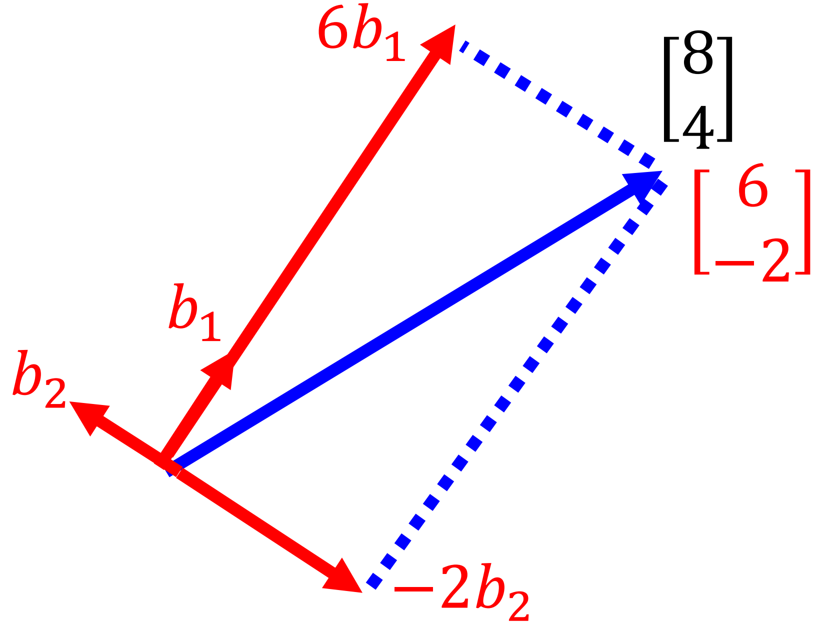
Coordinate System

Using Basis to represent Vector

Vector

$b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

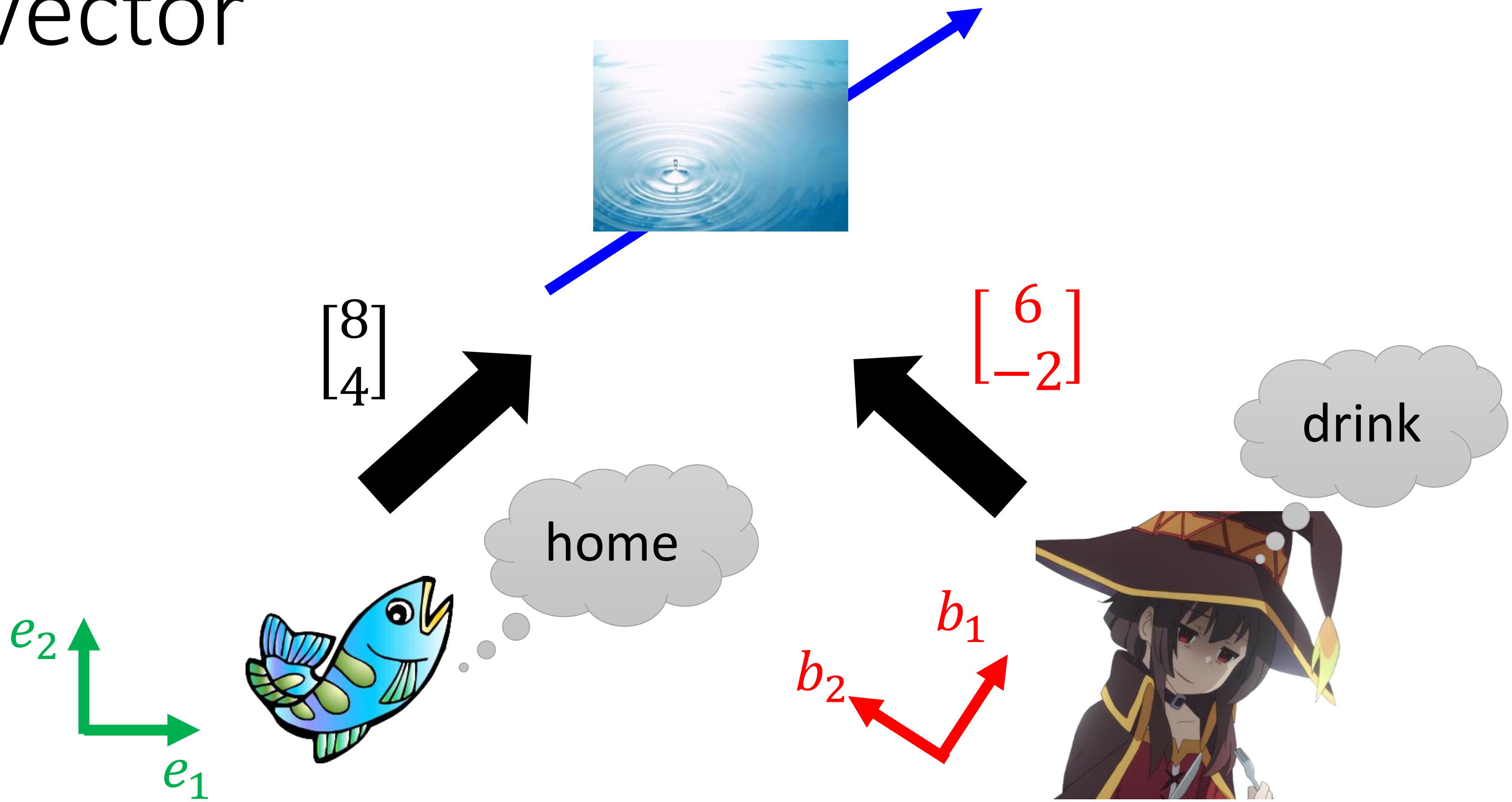




$\begin{bmatrix} 8 \\ 4 \end{bmatrix} = 8e_1 + 4e_2$

$\begin{bmatrix} 8 \\ 4 \end{bmatrix} = 6b_1 + (-2)b_2$

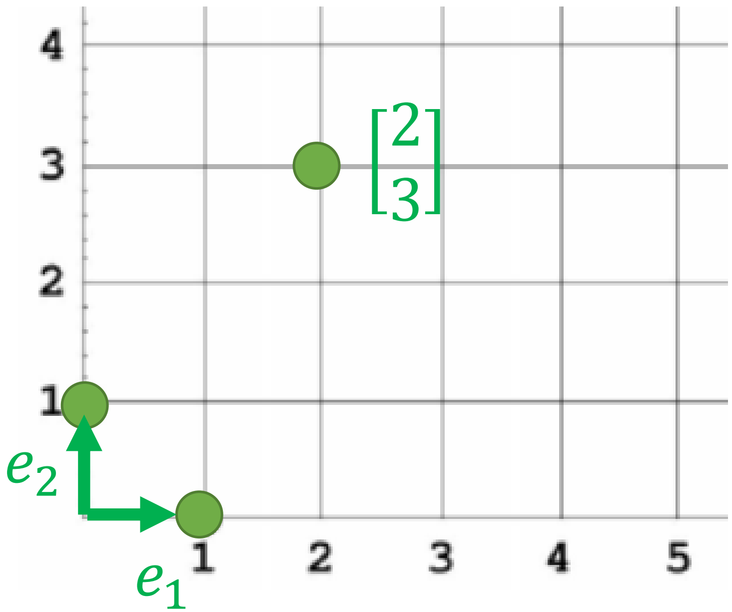
Vector

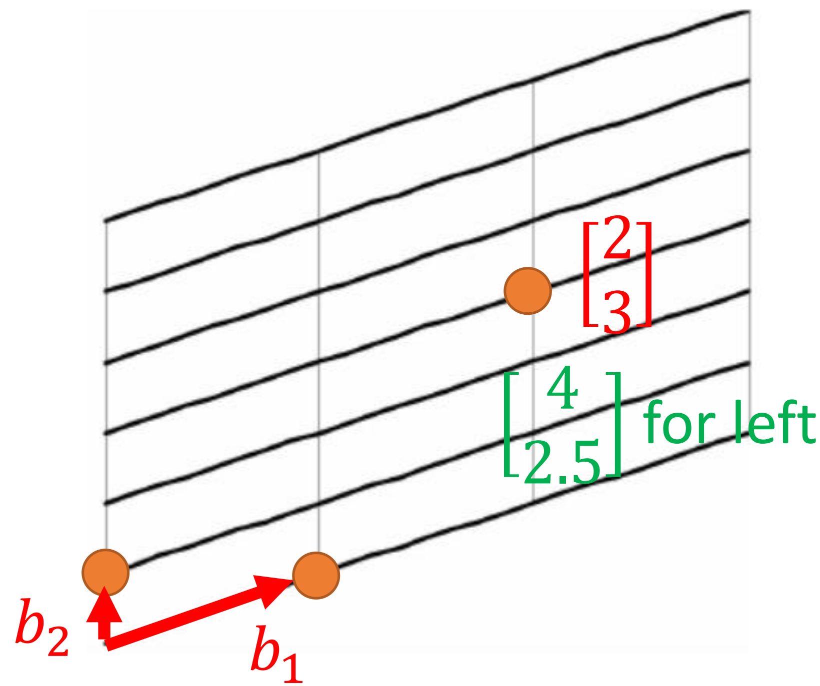


$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Vector

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $b_1 = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$ $b_2 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$



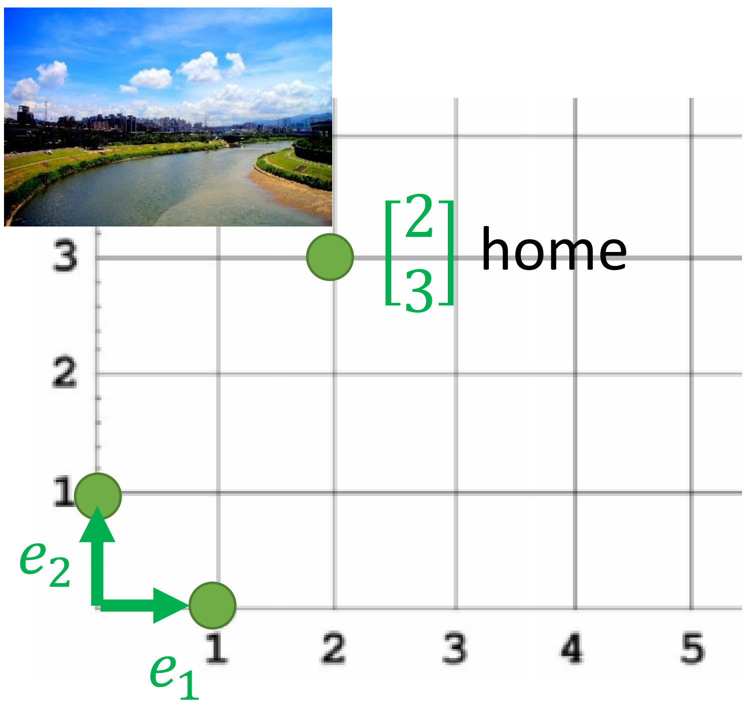


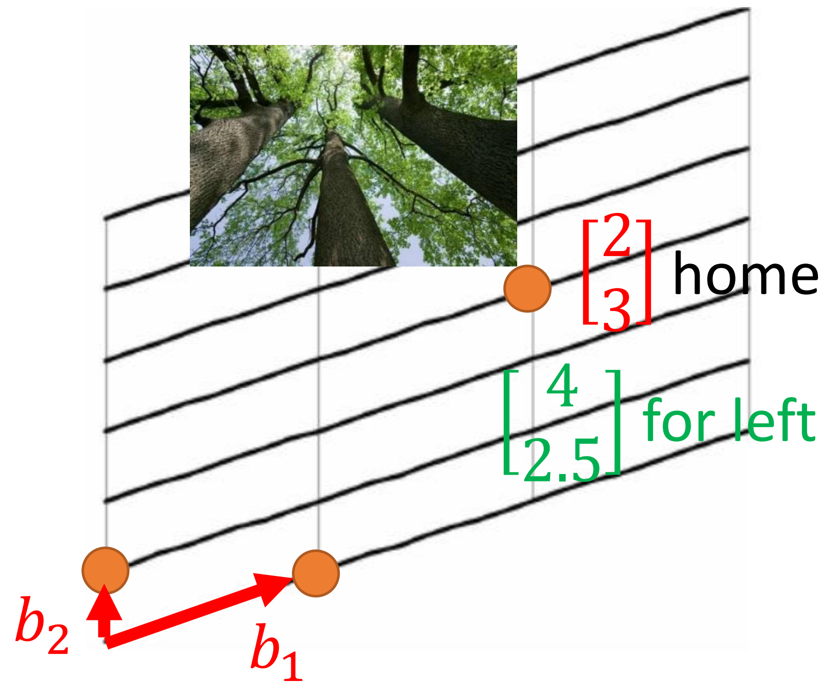
$$2b_1 + 3b_2 = \begin{bmatrix} 4 \\ 2.5 \end{bmatrix}$$

Vector

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$b_1 = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$ $b_2 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$





$2b_1 + 3b_2 = \begin{bmatrix} 4 \\ 2.5 \end{bmatrix}$

Coordinate System

- A vector set $\mathcal{B}$ can be considered as a coordinate system for R^n if:
- 1. The vector set $\mathcal{B}$ spans the R^n

$\Rightarrow$ Every vector should have representation

- 2. The vector set $\mathcal{B}$ is independent

$\Rightarrow$ Unique representation

$\mathcal{B}$ is a basis of R^n

Why Basis?

- Let vector set $\mathscr{B}=\{u_1, u_2, \cdots, u_k\}$ be independent.
- Any vector v in Span $\mathscr{B}$ can be uniquely represented as a linear combination of the vectors in $\mathscr{B}$.
- That is, there are unique scalars $a_1, a_2, \cdots, a_k$ such that $v = a_1u_1 + a_2u_2 + \cdots + a_ku_k$
- Proof:

Unique? $v = a_1u_1 + a_2u_2 + \cdots + a_ku_k$

$$v = b_1u_1 + b_2u_2 + \cdots + b_ku_k$$

$$(a_1 - b_1)u_1 + (a_2 - b_2)u_2 + \cdots + (a_k - b_k)u_k = 0$$

$\mathscr{B}$ is independent $\Rightarrow$ $a_1 - b_1 = a_2 - b_2 = \cdots = a_k - b_k = 0$

Coordinate System

- Let vector set $\mathcal{B}$=$\{u_1, u_2, \cdots, u_n\}$ be a basis for a subspace R^n

$\Rightarrow$ $\mathcal{B}$ is a coordinate system

- For any v in R^n, there are unique scalars $c_1, c_2, \cdots, c_n$ such that $v = c_1u_1 + c_2u_2 + \cdots + c_nu_n$

$\mathcal{B}$ -coordinate vector of v:

$$[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathcal{R}^n$$

(用 $\mathcal{B}$ 的觀點來看原來的 v)

Coordinate System

vector $\longrightarrow$ $\mathcal{B}=\{u_1, u_2, \cdots, u_n\}$

$\overbrace{[\underline{v}]_{\mathcal{B}}}^{\text{vector}}$

vector $\longrightarrow$ $\mathcal{E}=\{e_1, e_2, \cdots, e_n\}$

(standard vectors)

$\mathcal{E}$ is Cartesian coordinate system (直角坐標系)

$$v = [v]_{\mathcal{E}}$$

Other System → Cartesian

$$\mathcal{B}=\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\2\\2\end{bmatrix}\right\} \quad [v]_{\mathcal{B}}=\begin{bmatrix}3\\6\\-2\end{bmatrix}$$

$$v = 3\begin{bmatrix}1\\1\\1\end{bmatrix}+6\begin{bmatrix}1\\-1\\1\end{bmatrix}-2\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}7\\-7\\5\end{bmatrix}$$

$$\mathcal{C}=\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix},\begin{bmatrix}7\\8\\9\end{bmatrix}\right\} \quad [u]_{\mathcal{C}}=\begin{bmatrix}3\\6\\-2\end{bmatrix}$$

$$u = 3\begin{bmatrix}1\\2\\3\end{bmatrix}+6\begin{bmatrix}4\\5\\6\end{bmatrix}-2\begin{bmatrix}7\\8\\9\end{bmatrix} = \begin{bmatrix}13\\20\\27\end{bmatrix}$$

Other System → Cartesian

- Let vector set $\mathcal{B}$=$\{u_1, u_2, \cdots, u_n\}$ be a basis for a subspace R^n
- Matrix $\mathrm{B} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$

Given $[v]_{\mathcal{B}}$, how to find v? $[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

$$v = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$$

$= B[v]_{\mathcal{B}}$ (matrix-vector product)

Cartesian → Other System

$$v = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix} \qquad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$$

find $[\mathbf{v}]_{\mathcal{B}}$

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix} \qquad [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

B is invertible (?)

independent

$$B[v]_{\mathcal{B}} = v \quad \Rightarrow \quad [v]_{\mathcal{B}} = B^{-1}v = \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix}$$

Cartesian ↔ Other System

- Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$

$$\begin{tikzcd} v \arrow[r, shift left, "{[v]_{\mathcal{B}} = B^{-1}v}"] & {[v]_{\mathcal{B}}} \arrow[l, shift left, "{v = B[v]_{\mathcal{B}}}"] \end{tikzcd}$$

$$= c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

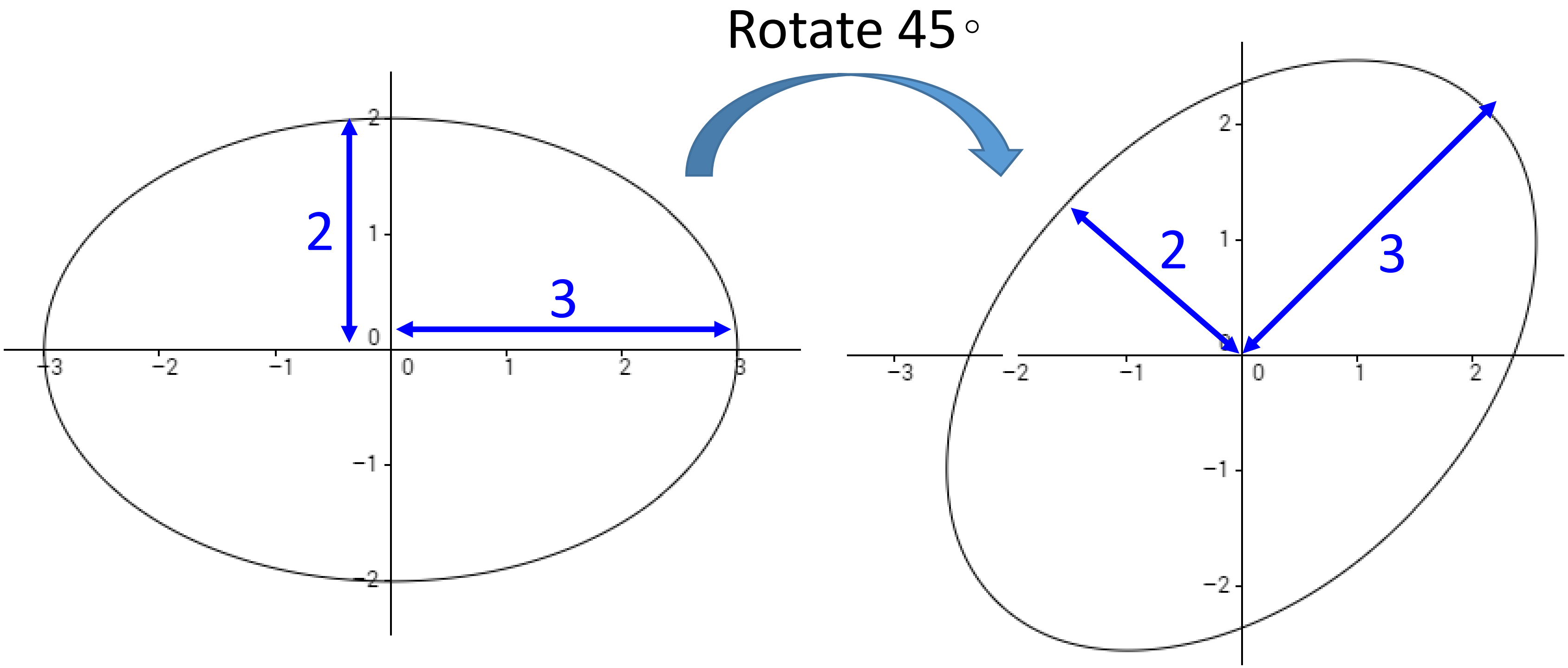
$$= \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Let $\mathcal{B}=\{b_1, b_2, \cdots, b_n\}$ is a basis of R^n. $[b_i]_{\mathcal{B}} =?$ e_i

(Standard vector)

Changing Coordinates

Equation of ellipse



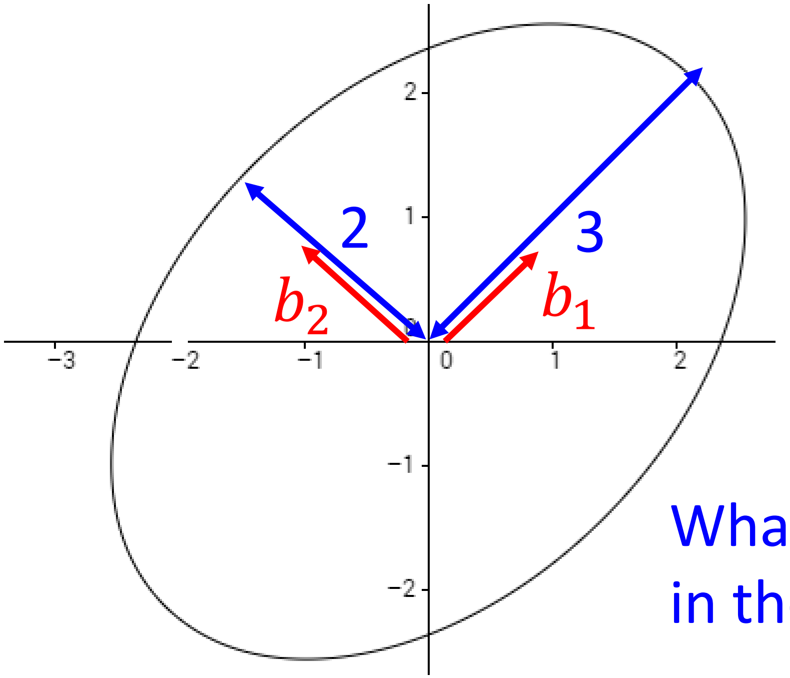
$$\frac{x^2}{3^2}+\frac{y^2}{2^2}=1$$

?

Equation of ellipse

Use another coordinate system

$$\mathcal{B} = \left\{ \overset{b_1}{\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}}, \overset{b_2}{\begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}} \right\}$$



$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}, [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

What is the equation of the ellipse in the new coordinate system?

$$\frac{(x')^2}{3^2} + \frac{(y')^2}{2^2} = 1$$

Equation of ellipse

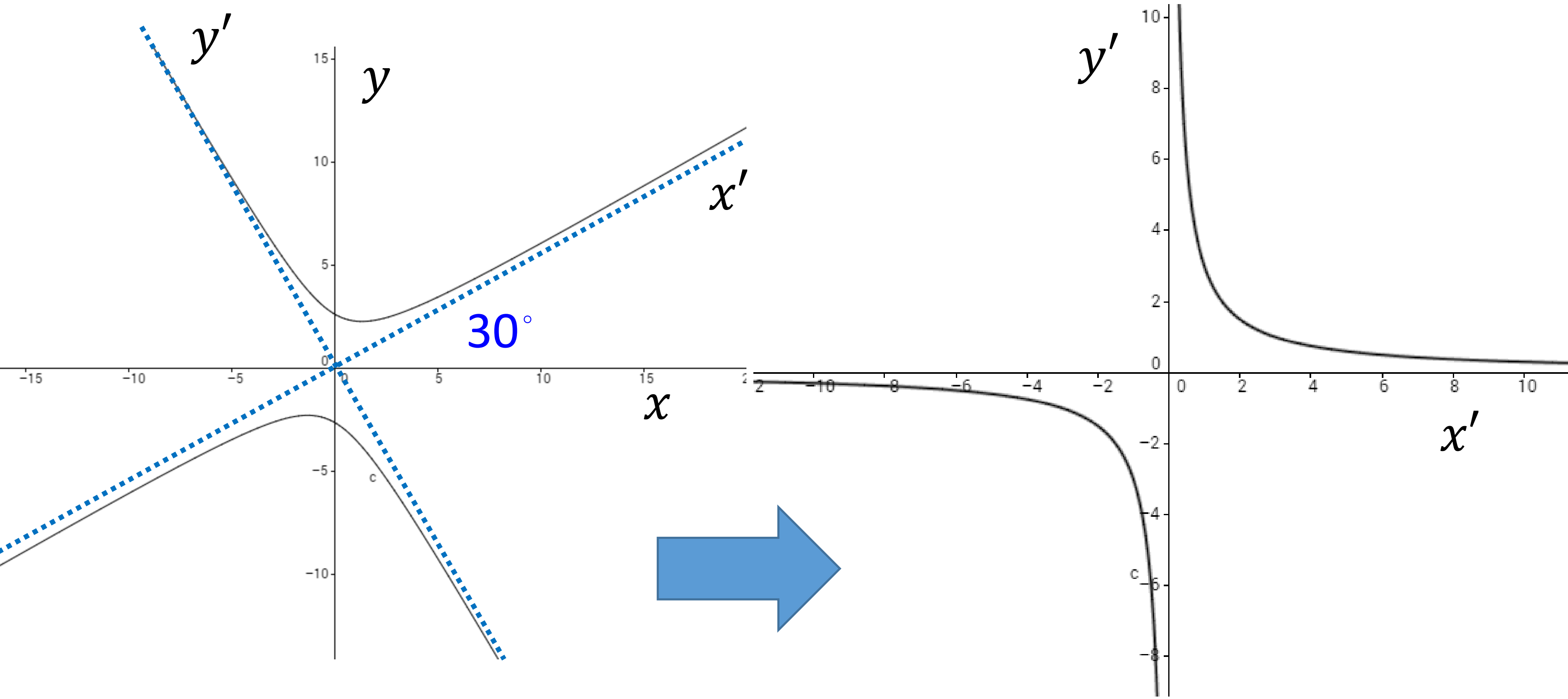
$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix},\ [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad \mathcal{B} = \left\{ \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \right\}$$

$$\frac{(x')^2}{3^2} + \frac{(y')^2}{2^2} = 1 \Rightarrow \frac{\left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)^2}{3^2} + \frac{\left(-\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)^2}{2^2} = 1$$

$$[\mathbf{v}]_{\mathcal{B}} = B^{-1}\mathbf{v}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = B^{-1}\begin{bmatrix} x \\ y \end{bmatrix}$$

Equation of hyperbola

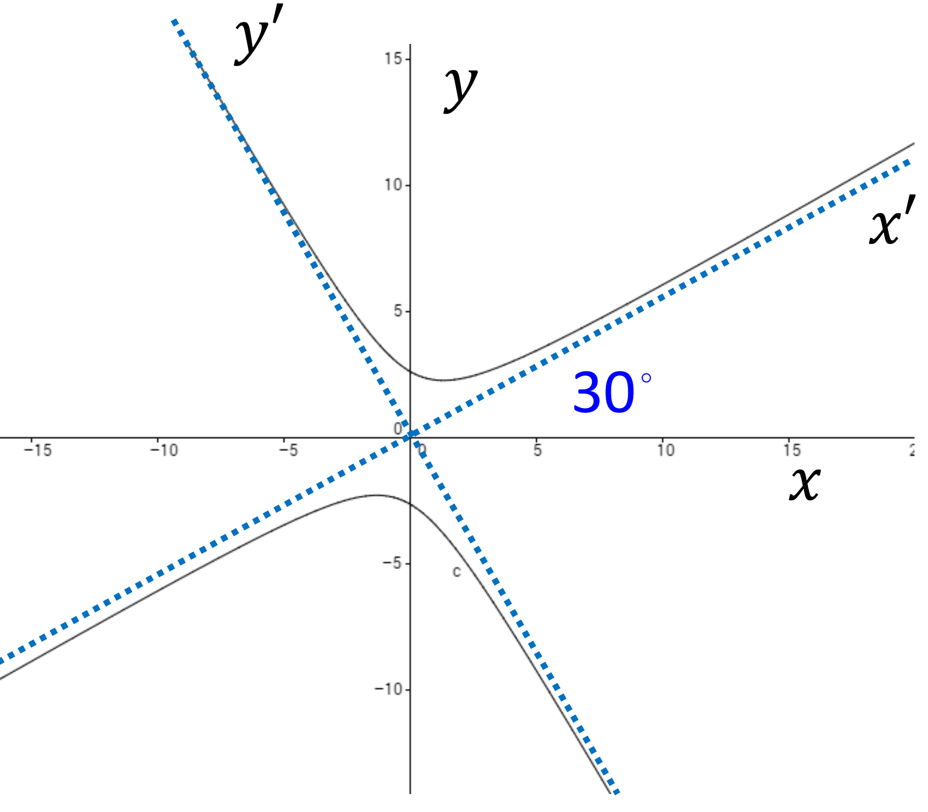


$$-\sqrt{3}x^2 + 2xy + \sqrt{3}y^2 = 12$$

$$equation?$$

Equation of hyperbola

$$B = [b_1 \quad b_2]$$



$$b_1 = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \quad b_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$v = \begin{bmatrix} x \\ y \end{bmatrix} \quad [v]_{\mathcal{B}} = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad v = \mathrm{B}[v]_{\mathcal{B}}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$-\sqrt{3}x^2 + 2xy + \sqrt{3}y^2 = 12$$

Equation of hyperbola

$$B = [b_1 \quad b_2]$$

$$-\sqrt{3}x^2 + 2xy + \sqrt{3}y^2 = 12$$

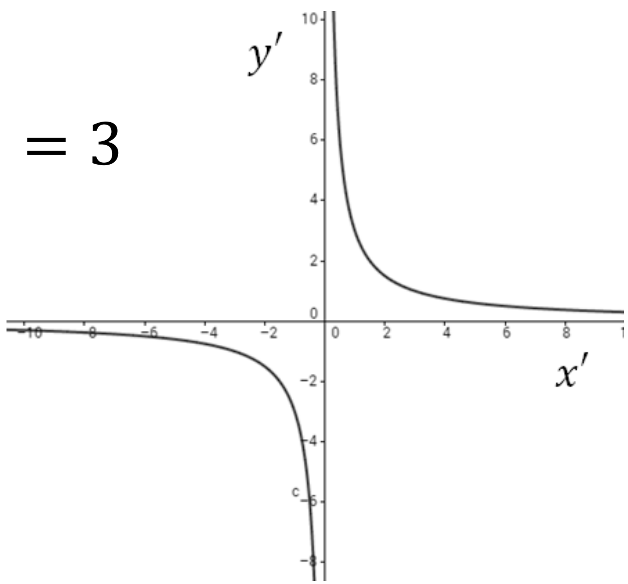
$$b_1 = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \quad b_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$x = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y'$$

$$y = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y'$$

$$v = \begin{bmatrix} x \\ y \end{bmatrix} \quad [v]_{\mathcal{B}} = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad v = \mathrm{B}[v]_{\mathcal{B}}$$

$$x'y' = 3$$



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Summary

$$\overbrace{[v]_{\mathcal{B}}}^{\text{vector}} \qquad \text{vector} \longrightarrow \mathcal{B} = \{u_1, u_2, \cdots, u_n\}$$

$$v \longrightarrow \text{vector} \longrightarrow \mathcal{E} = \{e_1, e_2, \cdots, e_n\}$$

(standard vectors)

$$v \quad \underset{v = B[v]_{\mathcal{B}}}{\overset{[v]_{\mathcal{B}} = B^{-1}v}{\rightleftarrows}} \quad [v]_{\mathcal{B}}$$

Appendix

Linear Algebra Review– Section 4.4 in one page

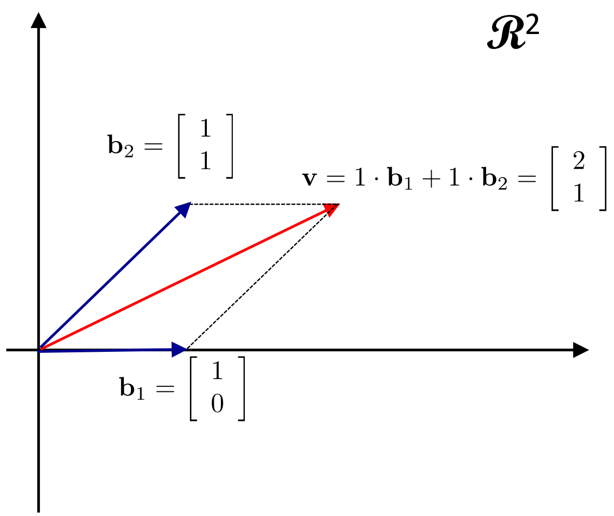
Given a basis $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ for the vector space $\mathscr{R}^n$.

Thm 4.10: Any vector $\mathbf{v} \in \mathcal{R}^n$ can be uniquely expressed as a linear combination of $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n$. That is,

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

Definition: $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}^T$ is called the **coordinate vector** of $\mathbf{v}$ relative to $\mathcal{B}$, or the **$\mathcal{B}$-coordinate vector** of $\mathbf{v}$.

Thm 4.11: $[\mathbf{v}]_{\mathcal{B}} = B^{-1}\mathbf{v}$ where $B \triangleq \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$.

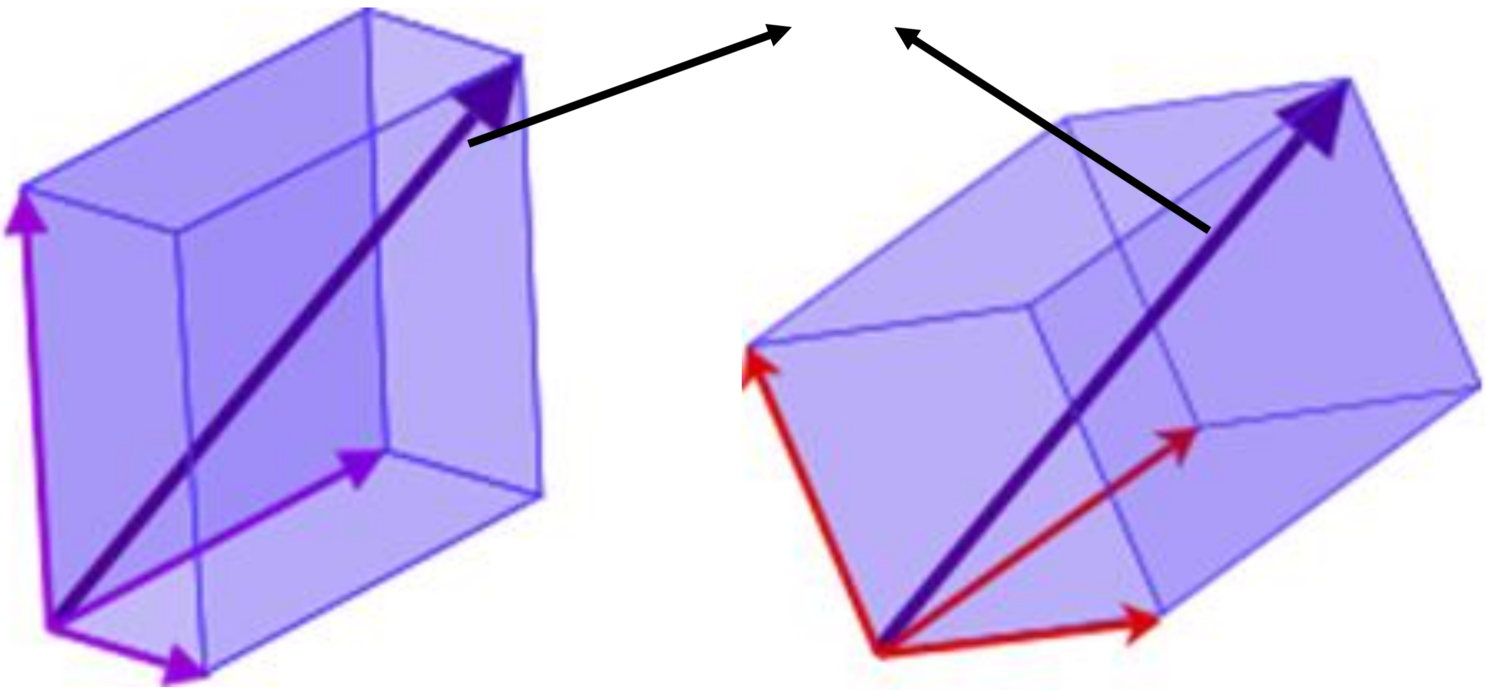


$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow B^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$[\mathbf{v}]_{\mathcal{B}} = B^{-1}\mathbf{v} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Vector

The same vector



Represented by purple arrows

Represented by red arrows